A STUDY OF CERTAIN CLASS OF DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER AND THEIR APPLICATIONS IN PHYSICAL PROBLEMS

SYNOPSIS

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CERTIFICATE

This is to certify that this Synopsis entitled **"A Study of Certain Class of Differential Equations of Fractional Order and Their Applications in Physical Problems"** by **Arpita Gupta,** submitted in fulfillment of the requirement for registration in Ph.D. Program in Department of Mathematics under Faculty of Science of JECRC University, Jaipur, during the academic year 2019, is a bonafid erecord work carried out under our guidance and supervision.

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1.Introduction

Before Newton and Leibnitz, the word calculus referred to anybody of mathematics but in the recent years calculus became animportant aspect in the field of mathematics. The concept of derivative is the principal origin for the enlargement of the calculus. Derivative describes the rate of change of a function, based on their geometrical or physical meaning. For example, the first and second order derivative of displacement w.r.t. time narrates the velocity in addition acceleration. In the same way, the third and fourth order derivative of displacement w.r.t. time is known as jerk and jounce.

Fractional calculus (FC) is one of the important branch of mathematical science in which we generalize the concept of differentiation and integration from integer order differentiation to arbitrary (real/complex) order differentiation, or in another way we can say that the derivative $\frac{d^n y}{dx^n}$ of integer order $n > 0$ be extended to *n* any order fractional fraction irrational or complex.

Since we all know that the function having first order derivative yields the slope, but what about if the function has half order derivative? The action of finding answers to such kinds of questions opened a new way and challenge to mathematician working in this field and real world problems. Thus the field of mathematics which discuss the derivative and integrals of arbitrary order of function is familiar as FC. ADE of fractional order (FO) is the advancement of classical DE by changing the classical order derivatives to arbitrary order derivatives.

First of all, De L. Hospital wrote a letter on 30-09-1695 to Leibnitz asking about the nth order derivative $\frac{d^n f(x)}{dx^n}$ of $f(x) = x$ at $n = 1/2$, then Leibnitz responded that " It will lead to a paradox and half order derivative of $f(x) = x$ is equal to $x\sqrt{(dx/x)}$".

Fundamentaldefinitions of FC

Some standard and fundamental definitions are given as follows

Caputo Fractional Derivative [1967]

The mathematician Riemann and Liouville were unable to give the information and modeling of certain complex curiosity. Then Caputo introduced a new derivative in 1967.

The Caputo derivative of FO$(n - 1 \leq \beta < n)$ of function $\psi(u)$ given as

$$ {}_{a}^{c}D_{u}^{\beta}\psi(u) = \frac{1}{\Gamma(n-\beta)}\int_{a}^{u}\frac{D^{n}\psi(\xi)}{(u-\xi)^{\beta-n+1}}d\xi \,, \tag{1}$$

where D^n denotes the n[th] order derivative of function and c stands for the Caputo term.

Grunwald-Letnikov (GL) Derivative [1998]

Let $y = \psi(u)$ be a continuous function then by the familiar definition of derivative we can find the first order derivative of $\psi(u)$ as-

$$\psi'(u) = \frac{d\psi}{du} = \lim_{h\to 0}\frac{\psi(u)-\psi(u-h)}{h}, \tag{2}$$

Derivative of GL type give the modification of classical order derivative to the FO derivative. Hence it can be represented in the following way

$$D^{\beta}\psi(u) = \lim_{h\to 0}\frac{1}{h^{\beta}}\sum_{m=0}^{\infty}(-1)^{m}\binom{\beta}{m}\psi(u-mh), \tag{3}$$

where $m \in \mathbb{N}$ and the binomial coefficient is calculated as

$$\binom{\beta}{m} = \frac{\beta(\beta-1)(\beta-2)\dots(\beta-m+1)}{m!}.$$

Riemann-Liouville (RL) Derivatives [1998]

For fixed $n \geq 1$ and integer $k \geq n$ the $(k-n)^{\text{th}}$ derivative of $\psi(u)$is specified as

$$\psi^{(k-n)}(u) = \frac{1}{\Gamma(n)}D^{k}\int_{a}^{u}(u-\xi)^{n-1}\psi(\xi)d\xi, \tag{4}$$

where the symbol $D^k(k \geq 0)$denotes *k* integrated differentiation.

For example:-With the help of above definition of arbitrary order derivative we can calculate the RL arbitrary order derivative ${}_{a}D_{u}^{\beta}\psi(u)$ of the function $\psi(u) = (u-a)^{\upsilon}$ as-

$${}_{a}D_{u}^{\beta}\psi((u-a)^{\upsilon}) = \frac{\Gamma(1+\upsilon)}{\Gamma(1+\upsilon-\beta)}(u-a)^{\upsilon-\beta}. \tag{5}$$

Caputo-Fabrizio (CF) Derivative [2015]

Let $\psi \in H'(l_1,l_2), l_2 > l_1, \beta \in [0,1]$, then CF derivative is defined as follows

$$D_{u}^{\beta}\big(\psi(u)\big) = \frac{M(\beta)}{1-\beta}\int_{a}^{u}\psi'(\xi)\exp\left[-\beta\frac{u-\xi}{1-\beta}\right]d\xi \;, 0 < \beta \leq 1 \tag{6}$$

$$= \frac{d\psi}{du} \;, \beta = 1 \,.$$

Atangana–Baleanu(AB) Derivative [2016]

Definition 1: Assume $\psi \in H^{'}(a,b), b > a, \beta \in [0,1]$ then, AB derivative in Caputo's type is given as

$$ {}^{ABC}_{a}D_{u}^{\beta}\left(\psi(u)\right) = \frac{M(\beta)}{1-\beta}\int_{a}^{u}\psi'(\xi)E_{\beta}\left[-\beta\frac{(u-\xi)^{\beta}}{1-\beta}\right]d\xi. \quad (7) $$

Here M(β)has the similar characteristics as in CF case.

Definition 2: Assume $\psi \in H^{'}(a,b), b > a, \beta \in [0,1]$ and differentiation of ψ is not possible then, AB derivative in RL type is represented as

$$ {}^{ABR}_{a}D_{u}^{\beta}\left(\psi(u)\right) = \frac{M(\beta)}{1-\beta}\frac{d}{dt}\int_{a}^{u}\psi(\xi)E_{\beta}\left[-\beta\frac{(u-\xi)^{\beta}}{1-\beta}\right]d\xi. \quad (8) $$

The AB derivative is very helpful to solve real world problem.

2. Review of Literature

In past some years differential equations (DE) of non integer order have been investigated by several mathematicians. The recognized books by Oldham and Spainer (1974), Miller and Ross (1993), Podlubny (1998) and Kilbas, et. al. (2006), on FC gives a brilliant account of its development and use in assorted fields of science and technology. Fractional differential equations (FDE) provides boundless and versatile application in engineering, sciences and real world problems, like in electromagnetic, viscoelasticity, fluid mechanics, electrochemistry, biological population model, optics, food sciences, rheology, acoustics, robotics, control theory, electrical and mechanical engineering, physics, chemistry, treatment of many diseases etc.

The application of non integer DE in physical problem or real life problem is studied by many mathematicians. So name a few are as methods, results and solutions of problems on DE of arbitrary order are given by Kilbas and Trujillo (2001). They discussed the technique focused on the reduction of Cauchy sense problem for the DE of arbitrary order to Volterra integral equations. Moreover the solutions of linear DE of FO are shown by the Laplace method, compositional methods and operational calculus. History, definition and application of FCfor the engineer are given by Loverro (2004). In this article, Loverro explained the utilization of FC in several physical problems. Solution of Fornberg–Whitham equation of

arbitrary order is given by Singh et. al (2012). To give the solution of such type of equation a newly technique namely homotopy perturbation transform method (HPTM)was discussed by Singh et. al (2012). The HPTM is a combined form of, homotopy perturbation method (HPM) [He (1999, 2000, 2006)], Laplace transform (LT) and He's polynomial. The reaction diffusion equations of FO were solved by Singh et. al (2013). In this paper, to obtain the solution of such kind of equation of complex order, RL derivative of arbitrary order as the time derivatives and as the space derivative Riesz-Feller fractional derivative were used. A groundwater flow model is proposed by Atangana andBildik (2013). In this paper, ground water flow equation of thiem and the theis type was converted to groundwater flow model of time fractional.Analytic solution of groundwater flow equation of thiemin addition the theis type was given by Laplace Carson transform method. A new solution of FDE using complex transform is introduced by Ibrahim and Darus (2014).To formulate new exact solutions for some non linearDE of arbitrary order these authors used the mobious method of DE of non integer order. Polar bear hairs explained by He et. al (2015). With the help of variational iteration method a new arbitrary order derivative defined. Moreover, applications were given in describing the thermal protection of polar bear hairs is explained. This paper has much importance in petroleum technology. A technique for solving heat like physical models in fractional order is presented by Khader et. al.(2015).A model of fluid flow with non integer derivative through porous media is given by Choudhary et. al (2016). Solution of the problem discussed in this research work is obtained by applying of the Fourier sine technique and sumudu technique. Smoking dynamics model is proposed by Singh et. al (2017). Using the Fixed point (FP) concept this paper discussed the existence of solution. A fractional model for damped Berger's equation is given by Singh et. al (2017). This model used in sense of CF fractional derivative and by using the concept of an iterative technique solution of the problem is obtained. For generalized casson fluid a comparative study of AB (Atangana-Baleanu) derivative in addition CF derivative was presented by Sheikh et. al (2017). In this article, AB fractional derivative and CF derivative are applied to free convective flow of generalized Casson fluid, exact solutions were obtained in to both the cases of AB derivative and CF derivative additionally compared the results graphically and also in tabular form. Non linear population growth model is proposed by Alkahtani et. al (2017). In this model for the kernel of differentiation, authors employed the generalized Mittag-Leffer function. Rubella

disease model with nonsingular fractional derivative is given by Koca (2017). Using the banach FP theorem authorselaborated the existence and uniqueness (EU) of exact solution of the studied problem.Application to MHD channel flow of nanofluids of AB fractional derivative was proposed by Saqib et. al (2018), Epidemiological model of arbitrary order for computer viruses is given by Singh et. al (2018). In this paper,authors discussed the computer viruses problem and used an iterative scheme to give solution of such kind of problem. Existence and uniqueness (EU) of the solution examined and verified. A diabetes model is given by Singh et. al(2018). This paper discussed the complications of diabetes with the help of CF (Caputo-Fabrizio) derivative, and obtained the analytical solution by homotopy analysis technique, LT scheme in addition pade approximation. Generalized fractional Schrodinger equations in quantum mechanics is proposed by Kumar et.al (2018). Solution of generalized fractional Schrodinger equation is given by using fractional Fourier transforms (FFT), inverse Fourier transforms (IFT) additionally LT. The solution of non linear optimization problem having CF derivative is given by Evirgen and Yavuz (2018).To obtain the optimal solution for the class of non linear programming problem, steepest descent method is used. Biological model of dengue is introduced by Jain and Atangana (2018). In this article, classical model is modified to new model of arbitrary order by using the concept of CF derivative. This new model analyzes the EU of the system solution. Uniformly differentiable functions (UDF) of non integer order are discussed by Kachhia and Prajapati (2018). This paper gives a novel idea of UDFof arbitrary order on an interval I (arbitrary interval) of R by applying the CF derivative.Analytical scheme to Navier–Stokes equation of non-integer order is discussed by Bairwa and Singh (2018). Iterative LTscheme is used for solvingnon integerNavier stokes equations in cylindrical coordinates.SIRS-SI malaria disease model with fractional order is given by Kumar et. al (2019). This model suggests the usages of vaccines and medicines of anti malarial. The existence of solution of SIRS-SI model of fractional order is checked by utilizing the FP concept. The numerical results of this model tell that the suggested technique is very authentic and impressive. AB non-integer order model to study the flow of Jeffrey nanofluid is studied by Ali et. al (2019). In this paper, under the effect of corresponding dimensionless parameters behavior of the fluid flow has shown.

3. Description of Topic

A study of certain class of DE of FO and their application in physical problem is our topic of research. Here we will discuss about important fractional derivatives. We will solve fractional differential equation (FDE) arising in the real world problem by numerical and analytical techniques. We will find out the application of FDE in physical problems likes problems arises in physics, chemistry, bioengineer, food sciences, mechanics etc.

4.Objectives of Research

Themain objectives of the proposed research work are as below

- To study linear and non-linear FDE arising in real word problems such as fractional Korteweg-de-Vries (KdV) equation, fractional Zakharov-Kuznetsov (ZK) equation, fractional Schrodinger equation, fractional Navier-Stokes equations, fractional Klein-Gordon (KG) equation, fractional thermal protection model for polar bear hairs, fractional groundwater flow equation, fractional reaction-diffusion equation, fractional diffusion equation, fractional wave equation, etc.
- To develop the analytical and numerical technique for solving linear DE and non-linear DE of arbitrary order.
- To examine application of FDE in linear and non linear sense.
- To study physical problems associated with DE of non integer order and then solve it applying distinct numerical and analytical methods.
- To study better physical behavior of FDE in real world problems such as fractional Korteweg-de-Vries (KdV) equation, fractional Zakharov-Kuznetsov (ZK) equation, fractional Schrodinger equation, fractional Navier-Stokes equations, fractional Klein-Gordon (KG) equation, fractional thermal protection model for polar bear hairs, fractional groundwater flow equation, fractional reaction-diffusion equation, fractional diffusion equation, fractional wave equation etc.
- **5. Research Methodology to be adopted**

  We will consider a certain class of linear in addition non-linear FDE arising in physical problem. We will demonstrate some new applications of FDE in real world problems. For this we will use numerical and analytical techniques. We will solve the non integer

differential equation by applying the strong analytic and numerical techniques such as HPM [He (1999, 2000, 2006)], HPM is a combo of the perturbation and homotopy method. This approach is very effective to solve the non linear DE without converting it into linear DE also it reduce the computational work and give very effective result, HPTM [Singh et. al (2012),Kumar et. al (2014)], HPTM is a consolidation of LT method, HPM and He's Polynomial. This technique is very impressive because He's polynomial handles the nonlinear terms very easily and escape the round off error, HAM [Liao (1995, 2003, 2012)], HAM is a semi analytical method which solves the nonlinear FDE and produce a convergent series solution. This scheme is very flexible to choose initial approximation and auxiliary linear operations, thus a complicated non linear problem can be transformed into an infinite number of simpler and linear sub problem, HATM [Singh (2018), Singh et. al (2019), Kumar et.al (2019)], this is a manipulation of HAM by using LT and it is very useful to approximate the solution of Volterra integral equation (VIE). This approach is very easy to implement and computationally very attractive, q-HATM [[Kumar et. al (2018), Singh et. al (2018)], this technique is very refined coupling of q-HAM and LT and give more precise and specific convergent series solution. To analyze non linear DE q-HATM is computationally very effective and authentic. We will also develop some powerful analyzed numerical techniques to solve fractional differential equation arising in physical problems.

We will use some software such as MATLAB, MAPLE, Mathematica for the numerical simulation.

6. Expected Outcome of the Research

The proposed study is focused to give the following expected outcomes

- Study of certain class of DE of arbitrary order of linear and non linear type will give very important and useful solutions to physical problems such as fractional Korteweg-de-Vries (KdV) equation, fractional Zakharov-Kuznetsov (ZK) equation, fractional Schrodinger equation, fractional Navier-Stokes equations, fractional Klein-Gordon (KG) equation, fractional thermal protection model for polar bear hairs, fractional groundwater flow equation, fractional reaction-diffusion equation, fractional diffusion equation, fractional wave equation, etc.

- This research work will provide the better physical description of real world problems such as fractional Korteweg-de-Vries (KdV) equation, fractional Zakharov-Kuznetsov (ZK) equation, fractional Schrodinger equation, fractional Navier-Stokes equations, fractional Klein-Gordon (KG) equation, fractional thermal protection model for polar bear hairs, fractional groundwater flow equation, fractional reaction-diffusion equation, fractional diffusion equation, fractional wave equation, etc.
- The proposed study will develop new analytical and numerical methods to solve FDE.
- The proposed study will give new direction of FDEappearing in scientific and technological field.

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